Question 1

(i) (a) $k = 20$. **B1** **1**

(i) (b) **M1** for attempt at u_3 term: $u_3 = k - \dfrac{36}{k-18}$ (at least this far)

A1 correct in a simplified form, at some stage: e.g. $\dfrac{k^2 - 18k - 36}{k - 18}$

M1 for equating their u_3 to 2 and creating and attempting to solve a polynomial in k

[$k^2 - 18k - 36 = 2k - 36$ or $k(k-20) = 0$]

Since $k \neq 20$, $k = 0$. **A1**

[*Condone lack of explanation for $k \neq 20$, but penalise if both answers offered.*]

NB $k = 0$ can be noted immediately from the shrewd observation that $\dfrac{a}{x}$ is a self-inverse function. This need not be stated explicitly, and candidates get all 4 marks. **4**

(i) (c) **M1** **A1** for $u_4 = \dfrac{k^3 - 18k^2 - 72k + 648}{k^2 - 18k - 36}$

M1 **A1** for $u_5 = \dfrac{k^4 - 18k^3 - 108k^2 + 1296k + 1296}{k^3 - 18k^2 - 72k + 648}$ [**M**'s are for single-fraction attempts]

M1 for equating their u_5 to 2 and creating and attempting to solve a polynomial in k

$$\frac{k^4 - 18k^3 - 108k^2 + 1296k + 1296}{k^3 - 18k^2 - 72k + 648} = 2 \qquad (\text{and } u_2 \neq 2,\ u_3 \neq 2)$$

A1 correct: $k^4 - 20k^3 - 72k^2 + 1440k = 0$ (must be simplified)

M1 for good factorisation attempt: $k(k-20)(k^2 - 72) = 0$ **A1** for $k = \pm 6\sqrt{2}$

B1 for explaining why $k \neq 0$, $k \neq 20$: these give sequence constant/ periodic with smaller Period 2. If they noted earlier that $u_2 \neq 2$, $u_3 \neq 2$, they earn this mark at that stage. **9**

(ii) If $u_n \geq 2$, then $u_{n+1} = 37 - \dfrac{36}{u_n} \geq 37 - \dfrac{36}{2} = 19 > 2$ **M1** (Must be a general process)

Since $u_1 = 2$, it follows that all terms of the sequence are ≥ 2 **A1** (convincing) **2**

Setting $u_{n+1} = u_n = l$ **M1** gives $l = 37 - \dfrac{36}{l} \Rightarrow 0 = l^2 - 37l + 36$ **A1** for quadratic

$\Rightarrow 0 = (l-1)(l-36)$ **M1** for solving: $l = 1$ or 36

Since the terms of the sequence are always greater than 1, $l = 36$

A1 Correct limit, from reasonable justification **4**

Question 2

$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ \ldots > \frac{8}{3}$ **M1** **A1** **2**

For $n = 4$, $4! = 24 > 16 = 2^4$ **B1**

For $n = 4 + k$, $n! = 4! \times 5 \times 6 \times \ldots\ldots \times (k+4) > 2^4 \times \underbrace{(2 \times 2 \times \ldots\ldots \times 2)}^{k \text{ times}} = 2^{4+k}$
since each following term in the factorial after the 4 is > 2 **B1** for a convincing explanation
OR
Using a more formal inductive proof: baseline check for $n = 4$ followed by

$k! > 2^k \Rightarrow (k+1)\,k! > (k+1)\,2^k > 2.2^k$ i.e. $(k+1)! > 2^{k+1}$ **M1** **A1** **2**

$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \sum_{n=4}^{\infty} \frac{1}{n!} = \frac{8}{3} + \sum_{n=4}^{\infty} \frac{1}{n!} < \frac{8}{3} + \sum_{n=4}^{\infty} \frac{1}{2^n}$

M1 for use of above result in e's series to create an inequality for the remaining terms
A1 correctly done (alternatives using different starting-points equally acceptable)

B1 for noting or using $\sum_{n=4}^{\infty} \frac{1}{2^n}$ is a GP with first term $\frac{1}{16}$ and common ratio $\frac{1}{2}$ (ditto)

[with $S_\infty = a / (1 - r) = \frac{1}{8}$] Hence $e < \frac{8}{3} + \frac{1}{8} = \frac{67}{24}$ **B1** **4**

$\frac{dy}{dx} = 6\,e^{2x} - \frac{14}{\frac{4}{3} - x}$ **B1**

M1 for testing value/sign of $\frac{dy}{dx}$ at $x = \frac{1}{2}$ and/or $x = 1$

When $x = \frac{1}{2}$, $\frac{dy}{dx} = 6e - 14 / \tfrac{5}{6} < \frac{67}{4} - \frac{84}{5} = 16.75 - 16.8 < 0$

dM1 for use of $e < \frac{67}{24}$ **A1** for correct conclusion (with correct number work)

When $x = 1$, $\frac{dy}{dx} = 6e^2 - 14 / \tfrac{1}{3} > 6 \times \frac{64}{9} - 42 = 42\tfrac{2}{3} - 42 > 0$

dM1 for use of $e > \frac{8}{3}$ **A1** for correct conclusion (w.c.n.w.)

Hence, by the "*Change of sign Rule*" (and continuity), y has a TP between $x = \frac{1}{2}$ and $x = 1$.

B1 for the explanation. And since $\frac{dy}{dx}$ goes –ve, 0, +ve, it is a MINIMUM TP **B1** **8**

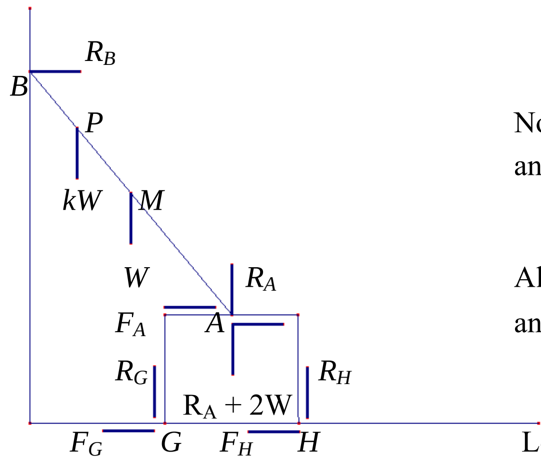
M1 for testing value/sign of $\frac{dy}{dx}$ at ($x = 1$ and) $x = \frac{5}{4}$ [$y'(1) > 0$ may be taken as read)

When $x = \frac{5}{4}$, $\frac{dy}{dx} = 6e^{2.5} - 14 / \tfrac{1}{12} < 6 \times 9\sqrt{3} - 168 < 6 \times 9 \times 2 - 168 = -60 < 0$

dM1 for use of $e < 3$ (e.g.) and $\sqrt{3} < 2$ (e.g.) **A1** for correct conclusion (w.c.n.w.)

Moreover, since $\frac{dy}{dx}$ goes +ve, 0, –ve, it is a MAXIMUM TP **B1** **4**

Question 9



Note that distance from A to table edge $= \frac{1}{4}a$

and height of table $= a$.

Also, $\mu_A = \frac{1}{2}$, $\mu_G = \mu_H = \frac{1}{3}$

and $\tan\theta = 2 \Rightarrow \sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = \frac{1}{\sqrt{5}}$.

Let $AP = \lambda a$ $(0 \le \lambda \le 6)$.

NOTES: Ignore use of an extra *g* in the weights – they will all cancel anyway

In moment equations, statements must actually be equations at some stage, and contain the correct number of forces (four if B↲ e.g.)

(i) R↑ for ladder $\quad R_A = (k+1)W$ **B1**

R→ for ladder $\quad R_B = F_A \le \frac{1}{2}(k+1)W$ **B1** + **B1** for *Friction Law* (inequality form)

A↲ for ladder $\quad W \,.\, 3a\cos\theta + kW \,.\, \lambda a\cos\theta = R_B \,.\, 6a\sin\theta$

M1 Must have 3 forces involved (×[d] by distances). sin/cos do not need to be correct or numerical at this stage

A1 correct, unsimplified $\quad \Rightarrow R_B = \frac{1}{12}(3+\lambda k)W$ **A1** correct, simplified

Eqlm. breaks down when $\frac{1}{12}(3+\lambda k)W = \frac{1}{2}(k+1)W$ **M1**

$\Rightarrow \lambda = 6 + \dfrac{3}{k}$ **A1** answer in useful form

Since $\lambda \le 6$ for painter on ladder, and $\lambda = 6 + \dfrac{3}{k} > 6$ for all positive *k*, the ladder does not slip on the table. **B1** for a good explanation of the result. **9**

OR Assuming painter is at the top of the ladder:

$R_A = (k+1)W$ **B1** $\quad R_B = F_A \le \frac{1}{2}(k+1)W$ **B1** + **B1** for *Friction Law* (≠ form)

A↲ $W \,.\, 3a\cos\theta + kW \,.\, 6a\cos\theta = R_B \,.\, 6a\sin\theta$ **M1** **A1** correct, unsimplified

$\Rightarrow R_B = \frac{1}{4}(1+2k)W$ **A1** correct, simplified

However, $F_{max} = \frac{1}{2}(k+1)W$ **M1**

Then **B2** for a convincing explanation that this is always less than R_B and that ladder does not slip **9**

(ii) R↑ for table $R_G + R_H = 12W$ **B1**

R→ for table $F_A = F_G + F_H \le \frac{1}{3}(R_G + R_H) = 4W$ **B1** + **B1** for *Friction Law*

Eqlm. broken by table slipping on ground when $\frac{1}{12}(3 + 9\lambda)W = 4W$ **M1** $\Rightarrow \lambda = 5$ **A1**

H↲ $12W \cdot \frac{1}{4}a = F_A \cdot a + R_G \cdot \frac{1}{2}a$

M1 Must have correct 3 forces involved ($\times^{d}$ by distances) **A1** correct, simplified

Eqlm. broken by turning about *H* when $R_G = 0$ **M1**

$\Rightarrow 3W = \frac{1}{4}W(1 + 3\lambda) \Rightarrow \lambda = 3\frac{2}{3}$ **A1**

Eqlm. is broken when table turns about *H* **B1**

when painter is a distance $\frac{11}{3}a$ up the ladder **B1**

[Correct conclusion; with reasons.] **11**

Question 10

(i) First collision:

$$\begin{array}{ccc} & A & B \\ \to u & \bullet \to v & \bullet \to w \\ & \downarrow & \downarrow \\ & m\text{ kg} & km\text{ kg} \end{array}$$

CLM: $mu = mv + kmw$ **B1**

NEL: $\frac{1}{2}u = w - v$ **B1**

M1 Solving $\Rightarrow$ $v = \dfrac{u(2-k)}{2(k+1)}$ **A1** and $w = \dfrac{3u}{2(k+1)}$ **A1** **5**

Second collision:

$$\begin{array}{ccc} & B & C \\ \to w & \bullet \to x & \bullet \to y \\ & \downarrow & \downarrow \\ & km\text{ kg} & 3m\text{ kg} \end{array}$$

CLM: $kmw = kmx + 3my$ **B1**

NEL: $\frac{1}{4}w = y - x$ **B1**

M1 Solving $\Rightarrow$ $x = \dfrac{w(4k-3)}{4(k+3)}$ or $\dfrac{3(4k-3)u}{8(k+1)(k+3)}$ **A1** N.B. y is not required. **4**

For second collision A & B, $v > x$ **M1** $\Rightarrow$ $\dfrac{u(2-k).4(k+3)}{8(k+1)(k+3)} > \dfrac{3(4k-3)u}{8(k+1)(k+3)}$

M1 Creating a quadratic inequality

$4(6 - k - k^2) > 12k - 9 \;\Rightarrow\; 0 > 4k^2 + 16k - 33 = (2k + 11)(2k - 3)$ **A1** correct

$\Rightarrow$ $0 > (2k+11)(2k-3)$ and since $k > 0$, we have $0 < k < \dfrac{3}{2}$ **A1**

Allow $k < \dfrac{3}{2}$ but not $-\dfrac{11}{2} < k < \dfrac{3}{2}$ **4**

(ii) When $k = 1$, $v = \frac{1}{4}u$, $w = \frac{3}{4}u$ and $x = \frac{3}{64}u$ **B1** FT all 3 from earlier results

Time taken from B to C is $\dfrac{4d}{3u}$ **B1**

during which time A travels a distance $\dfrac{u}{4}.\dfrac{4d}{3u} = \dfrac{d}{3}$ **B1**

Relative speed of A to B is $\dfrac{13u}{64}$ **B1**

Time taken for A to catch up B is $\dfrac{2d}{3} \div \dfrac{13u}{64} = \dfrac{128d}{39u}$ **M1** **A1**

Total time between collisions is thus $\dfrac{4d}{3u} + \dfrac{128d}{39u} = \dfrac{60d}{13u}$ **A1** gained legitimately **7**

Question 11

$x = ut\cos\theta - \frac{1}{2}ft^2$ **B1** $\quad y = ut\sin\theta - \frac{1}{2}gt^2$ **B1** Seen anywhere

$y = 0,\ t \neq 0$ when $t = \dfrac{2u\sin\theta}{g}$ **M1 A1**

Then $OA = u\,.\,\dfrac{2u\sin\theta}{g}\,.\cos\theta - \frac{1}{2}f\dfrac{4u^2\sin^2\theta}{g^2}$ **M1** $= \dfrac{2u^2\sin\theta}{g^2}(g\cos\theta - f\sin\theta)$ **A1** **6**

(i) $\dot{x} = 0$ when $t = \dfrac{u\cos\theta}{f}$ **M1 A1**

For blow-back to happen before landing, $\dfrac{u\cos\theta}{f} < \dfrac{2u\sin\theta}{g}$ **M1** (or equivalent alternative)

$\Rightarrow \tan\theta > \dfrac{g}{2f}$ and $\alpha = \arctan\left(\dfrac{g}{2f}\right)$ **A1** **4**

(ii) For second particle, same as *OA* but with $\theta = 45^\circ$:

$$OB = \frac{u^2\sqrt{2}}{g^2}(g - f)\frac{\sqrt{2}}{2} = \frac{u^2(g-f)}{g^2}$$ **B1** FT

OA maximised when $g\,2\sin\theta\cos\theta - f\,2\sin^2\theta$ is

$\equiv g\sin 2\theta - f(1 - \cos 2\theta)$ **M1** for double-angle use (or equivalent calculus)

$\equiv g\sin 2\theta + f\cos 2\theta - f$ **A1**

$\equiv \sqrt{f^2 + g^2}\cos(2\theta - \phi) - f$ for some ϕ **M1 A1**

$\leq \sqrt{f^2 + g^2} - f$ **A1**

Then $\dfrac{OB}{OA} = \dfrac{g - f}{\sqrt{f^2 + g^2} - f}$ (cancelling u^2/g^2) **M1 A1** **8**

When $f = g$, the resultant force is in the direction of the initial velocity **B1**

so that *B* moves up and down the straight line at 45° to the horizontal **B1** **2**

Question 12

(i) $p(1\text{ W}) = p(1, 0, 0) + p(0, 1, 0) + p(0, 0, 1)$ **B1**

$$= 30\left(\frac{1}{36}\right)\left(\frac{35}{36}\right)^{29}\left(\frac{24}{25}\right)^{30}\left(\frac{40}{41}\right)^{30} + \left(\frac{35}{36}\right)^{30} 30\left(\frac{1}{25}\right)\left(\frac{24}{25}\right)^{29}\left(\frac{40}{41}\right)^{30} + \left(\frac{35}{36}\right)^{30}\left(\frac{24}{25}\right)^{30} 30\left(\frac{1}{41}\right)\left(\frac{40}{41}\right)^{29}$$

M1 **A1** **A1** **A1**

$p(A = 1 \mid W = 1) = \left(\frac{N}{D}\right)$ where $N = p(1, 0, 0)$ and D = the above

M1 for suitable attempt at conditional probability **A1** for attempt to use of these results

$$N = \frac{30.24.40(35\times 24\times 40)^{29}}{(36\times 25\times 41)^{30}} \quad \text{and} \quad D = \frac{30(35\times 24\times 40)^{29}}{(36\times 25\times 41)^{30}}\left[24.40 + 35.40 + 35.24\right]$$

M1 for decent attempt to simplify **dM1** for cancelling multiple terms

$$\Rightarrow p(A = 1 \mid W = 1) = \frac{24\times 40}{24\times 40 + 35\times 40 + 35\times 24} = \frac{24\times 4}{24\times 4 + 35\times 4 + 7\times 12} = \frac{24}{24 + 35 + 7\times 3}$$

$$= \frac{24}{80} = \frac{3}{10}$$ **A1** **10**

(ii) Avge. (Expected) no. of wkts. is $30\left(\frac{1}{36} + \frac{1}{25} + \frac{1}{41}\right)$ **M1** $= \frac{5}{6} + \frac{6}{5} + \frac{30}{41} \approx 2 + \frac{3}{4} \approx 3$ **A1**

or via more careful working such as $\frac{61}{30} + \frac{30}{41} = \frac{2501 + 900}{1230} = \frac{3401}{1230} \approx 3$ etc. **2**

(iii) Binomial ≈ Poisson **B1** Since n is large and p is small **B1** **2**

$W \approx P_o(3)$ **M1** for use of this $p(W \geq 5) = 1 - \{p_0 + p_1 + p_2 + p_3 + p_4\}$ **M1**

$$= 1 - e^{-3}\left(1 + 3 + \frac{3^2}{2} + \frac{3^3}{6} + \frac{3^4}{24}\right)$$ **A1**

$$\approx 1 - \frac{1}{20}\left(1 + 3 + 4.5 + 4.5 + 3.375\right)$$ **A1**

(Or by use of tables) $= 1 - \frac{1}{20}\left(16.375\right)$ **A1** $\approx 1 - \frac{4}{5} = \frac{1}{5}$ **A1** **6**

Use of a Normal approximation (noting that $np < 5$) loses one mark:

$W \approx N(3, 2.9)$ **M1** **A1** $p(W \geq 5) = p(W_c > 4.5)$ **M1** (cont^y. correction)

$$= p\left(z > \frac{4.5 - 3}{\sqrt{2.9}}\right) \approx p\left(z > \frac{1.5}{1.7}\right) \approx p\left(z > 0.882\right)$$ **A1** awrt 0.88

$$= 1 - \Phi(0.882) \approx 1 - 0.8 = \frac{1}{5}$$ **A1**

Question 13

(i) For $n = 4$ …

Order	Choice	Order	Choice	Order	Choice	Order	Choice
1234	4	2134	1	3124	1	4123	1
1243	3	2143	1	3142	1	4132	1
1324	4	2314	1	3214	2	4213	2
1342	2	2341	1	3241	2	4231	2
1423	3	2413	1	3412	1	4312	3
1432	2	2431	1	3421	2	4321	3

M1 Breakdown into cases **A1** 4! = 24 of them **A1** Systematic **A1** Most "choices" correct

Thus $P_4(1) = \frac{11}{24}$ **A1** , $P_4(2) = \frac{7}{24}$ **A1** , $P_4(3) = \frac{4}{24}$ or $\frac{1}{6}$ **A1** , $P_4(1) = \frac{2}{24}$ or $\frac{1}{12}$ **A1** **8**

(ii) Let $N(k)$ denote the number of ways of getting the k^{th} largest ice-cream.

If the first ice-cream is the largest, then no larger is offered, and I am left with one of the remaining sizes of ice-cream, each equally likely. So, $N(1) = 0$ **B1**

If the first ice-cream is the 2nd largest, then I am certain to choose the largest when it is offered. So, $N(1) = (n-1)!$ **B1** Answer **B1** Explanation

If the first ice-cream is the third, then I choose whichever of 1 or 2 is offered first, with each possibility being equally likely. So, $N(1) = \frac{1}{2}(n-1)!$

B1 Answer **M1** **A1** Explanation

In the same way, if the first ice-cream is r , then each of the ice-creams $1, 2, \ldots, (r-1)$ are equally likely to be chosen. So, $N(1) = \frac{(n-1)!}{(r-1)}$ **B1** Answer **M1** **A1** Explanation

Altogether, then, $P_n(1) = \frac{1}{n!}\left\{0 + (n-1)! + \frac{1}{2}(n-1)! + \frac{1}{3}(n-1)! + \ldots.. + \frac{1}{n-1}(n-1)!\right\}$ **M2**

$$= \frac{1}{n}\left\{0 + 1 + \frac{1}{2} + \frac{1}{3} + \ldots.. + \frac{1}{n-1}\right\} \quad \text{or} \quad \frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{r}$$ **A1** **12**

For a correct answer with **no** explanation at all: **7**
For correct answer with a lack of a convincing general explanation: **9**